JOURNAL OF APPROXIMATION THEORY **31**, 288–297 (1981)

On Cardinal Exponential Splines of Higher Order

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Communicated by I. J. Schoenberg

Received July 23, 1980

1. INTRODUCTION

The notion of cardinal exponential spline interpolant (of order zero) has been introduced and thoroughly investigated by Schoenberg [9, 10]. In a recent paper [1], Greville *et al.* have extended this notion to the case of order $r \geqslant 1$ (cf. Section 4 below). Moreover, these authors have thoroughly studied their extension in terms of the shift operator. On the other hand, the author of the present paper has used an alternative approach to certain classes of cardinal spline functions, adopting the viewpoint of integral transform theory and complex analysis. Since this method is based on the notion of "discontinuous factor" it produces complex contour integral representations (with non-compact integration paths) of cardinal spline functions in a very natural way [4–6]. From these representations all the information that is needed may be obtained by the calculus of residues. It is the purpose of the present paper to establish in the same vein a complex contour integral representation of the cardinal exponential spline interpolants of the first order (Section 4). On the basis of this result we shall determine the pointwise convergence behaviour on $\mathbb{R}$ of the cardinal exponential spline interpolants of order $r \geqslant 0$ and successively higher degree (Section 5). For the reader's convenience, a résumé of the complex contour integral representation of cardinal exponential splines (of order zero) and the closely related Euler–Frobenius polynomials will be given in Sections 2 and 3, respectively. Finally, Section 6 summarizes some general principles concerning the application of integral transform techniques to the theory of (univariate) cardinal and periodic spline functions. The object is to emphasize the close connection of these classes of splines with the harmonic analysis.

0021-9045/81/030288–10$02.00/0

2. Cardinal Exponential Spline Functions

For any integer $m \geqslant 1$ let $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ denote the complex vector space of all cardinal spline functions of degree m on the real line $\mathbb{R}$ with respect to the bi-infinite knot sequence $\mathbb{Z}$ of integer points. If h denotes a complex number $\neq 0$ (we shall write $h \in \mathbb{C}^\times$ as a shorthand), the set of cardinal spline functions $s_m \in \mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ that satisfy the homogeneous linear difference equation of the first order

$$f(x+1) - hf(x) = 0 \qquad (x \in \mathbb{R}) \tag{1}$$

forms a one-dimensional vector subspace $\mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z})$ of $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$. The elements of the space $\mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z})$ are called cardinal exponential splines of degree $m \geqslant 1$ and weight h (Schoenberg [9, 10]). For weights h that do not belong to the unit circle $U = \{z \in \mathbb{C} \mid |z| = 1\}$ in the complex place $\mathbb{C}$ we may establish the following complex contour integral representation with non-compact path.

THEOREM 1. *Let the number $h \in \mathbb{C}^\times - U$ be fixed and let P denote the positively oriented boundary of any closed vertical strip in the open complex right, resp. left, half-plane that contains the line $\{z \in \mathbb{C} \mid \operatorname{Re} z = \log|h|\}$ in its interior. Then the vector subspace $\mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z})$ of $\mathfrak{S}_m(\mathbb{R}; \mathbb{Z})$ may be spanned by the cardinal exponential spline $S_{m,h}$ of degree $m \geqslant 1$ and weight h given by*

$$S_{m,h}\colon \mathbb{R} \ni x \rightsquigarrow \frac{1}{2\pi i} \int_P \frac{e^{(x+1)z}}{(e^z - h)\, z^{m+1}}\, dz, \tag{2}$$

i.e., $\mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z}) = \mathbb{C} \cdot S_{m,h}$.

The preceding theorem represents the main result of Ref. [4]. Its proof depends upon a complex line integral representation of the basis splines which may be established by an application of the inverse bilateral Laplace transform (cf. Section 6 infra).

3. Euler–Frobenius Polynomials

Let us keep to the above notations. For any number $h \in \mathbb{C}^\times$ there exists a cardinal exponential spline $s_m^{(0)} \in \mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z})$ of degree $m \geqslant 1$ and weight h that satisfies the interpolation condition

$$s_m^{(0)}(n) = h^n \qquad (n \in \mathbb{Z}) \tag{3}$$

if and only if $S_{m,h}(0) \neq 0$.

With this in mind define

$$p_m\colon h \rightsquigarrow m!\,\frac{(h-1)^{m+1}}{h}\,S_{m,h}(0) \qquad (m \geqslant 1). \tag{4}$$

Then we may establish (cf. [5]):

THEOREM 2. *For all integers $m \geqslant 1$, p_m is a monic polynomial of degree $m-1$ with strictly positive integer coefficients. It satisfies $p_m(0) = 1$.*

Following the terminology of Schoenberg [9, 10], $(p_m)_{m \geqslant 1}$ are called Euler–Frobenius polynomials. In view of (4) and (2) these polynomials admit the complex contour integral representation

$$p_m(h) = \frac{(h-1)^{m+1}}{h}\,\frac{m!}{2\pi i}\int_P \frac{e^z}{(e^z-h)\,z^{m+1}}\,dz \tag{5}$$

for $h \in \mathbb{C}^\times - U$. A short computation establishes the following:

THEOREM 3. *For any number $h \in \mathbb{C}^\times - U$ the first derivatives $(p'_m)_{m \geqslant 1}$ of the Euler–Frobenius polynomials admit the complex contour integral representation*

$$p'_m(h) = \frac{mh+1}{h(h-1)}\,p_m(h) + \frac{(h-1)^{m+1}}{h}\cdot\frac{m!}{2\pi i}\int_P \frac{e^z}{(e^z-h)^2\,z^{m+1}}\,dz, \tag{6}$$

where the non-compact path P of integration occurring in (6) is defined as in Theorem 1 supra.

From the complex contour integral representations (5) and (6) we are able to deduce the following:

COROLLARY 1. *The Euler–Frobenius polynomials $(p_m)_{m \geqslant 1}$ satisfy the three-term recurrence relation*

$$p_{m+1}(h) = (mh+1)\,p_m(h) - h(h-1)\,p'_m(h) \qquad (m \geqslant 1). \tag{7}$$

In particular we have $p_m(1) = m!$

Proof. For $h \in \mathbb{C}^\times - U$ let $(z_k(h))_{k \in \mathbb{Z}}$ denote the sequence of zeros of the entire holomorphic function $z \rightsquigarrow e^z - h$. An application of the calculus of residues furnishes the identities

$$\begin{aligned} \frac{m!}{2\pi i}\int_P \frac{e^z}{(e^z-h)^2\,z^{m+1}}\,dz &= -\frac{1}{h}\,(m+1)!\sum_{k \in \mathbb{Z}} \frac{1}{z_k^{m+2}(h)} \\ &= -\frac{1}{(h-1)^{m+2}}\,p_{m+1}(h). \end{aligned} \tag{8}$$

From (8) and (6) the recurrence relation (7) becomes obvious. ∎

In particular, the recursion formula (7) implies via the intermediate value theorem (cf. Quade and Collatz [2]):

COROLLARY 2. *For all integers $m \geqslant 2$, the roots of the Euler–Frobenius polynomials p_m are simple and located on the open negative real half-line $\mathbb{R}_-^\times$.*

It should be noted that the Euler–Frobenius polynomials play an important rôle in the theory of cardinal splines as well as in the theory of periodic splines. See, for instance, the paper [2] by Quade and Collatz cited above, and the forthcoming notes [8].

4. CARDINAL EXPONENTIAL SPLINE INTERPOLANTS OF HIGHER ORDER

Let $h \in \mathbb{C}^\times - U$ denote a weight so that $p_m(h) \neq 0$ for an integer $m \geqslant 1$. In view of (2) and (4) the cardinal exponential spline

$$s_m^{(0)}: \mathbb{R} \ni x \rightsquigarrow \frac{(h-1)^{m+1}}{hp_m(h)} \frac{m!}{2\pi i} \int_P \frac{e^{(x+1)z}}{(e^z - h)\, z^{m+1}}\, dz \tag{9}$$

is the unique element of the space $\mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z})$ that satisfies $s_m^{(0)}(0) = 1$ and hence the interpolation condition (3). Following Greville *et al.* [1], for any integer $r \geqslant 0$ the function on $\mathbb{R}$

$$s_m^{(r)} = \frac{1}{r!} \frac{\partial^r}{\partial h^r} s_m^{(0)} \qquad (m \geqslant 1) \tag{10}$$

will be called the cardinal exponential spline interpolant of degree m and order r with respect to the bi-infinite geometric sequence $(h^n)_{n \in \mathbb{Z}}$.

In the case $r = 1$ we conclude from (9) and the recurrence formula (7) the following result:

THEOREM 4. *Let the weight $h \in \mathbb{C}^\times - U$ satisfy $p_m(h) \neq 0$ for an integer $m \geqslant 1$. The cardinal exponential spline interpolant of degree m and of the first order with respect to the bi-infinite geometric sequence $(h^n)_{n \in \mathbb{Z}}$ admits the complex contour integral representation*

$$\begin{aligned} s_m^{(1)}: \mathbb{R} \ni x \rightsquigarrow & \frac{p_{m+1}(h)}{(h-1)\, p_m(h)} s_m^{(0)}(x-1) \\ & + \frac{(h-1)^{m+1}}{hp_m(h)} \frac{m!}{2\pi i} \int_P \frac{e^{(x+1)z}}{(e^z - h)^2\, z^{m+1}}\, dz. \end{aligned} \tag{11}$$

The path P of integration is defined as in Theorem 1 supra.

As a consequence we may establish the following reduction formula:

THEOREM 5. *For* $x \in \mathbb{R}$ *and all weights* $h \in \mathbb{C}^{\times} - U$ *so that* $p_m(h) \neq 0$ $(m \geqslant 1)$ *the identity*

$$s_m^{(1)}(x) = x s_m^{(0)}(x-1) - \frac{p_{m+1}(h)}{(h-1)\,p_m(h)} \left(s_{m+1}^{(0)}(x-1) - s_m^{(0)}(x-1)\right) \quad (12)$$

is obtained.

Proof. An application of Cauchy's residue theorem yields

$$\frac{1}{2\pi i} \int_P \frac{e^{(x+1)z}}{(e^z - h)^2 z^{m+1}}\, dz = \sum_{k \in \mathbb{Z}} \left[\frac{x e^{(x-1) z_k(h)}}{z_k^{m+1}(h)} - (m+1) \frac{e^{(x-1) z_k(h)}}{z_k^{m+2}(h)} \right] \quad (13)$$

for all $x \in \mathbb{R}$. In view of (9) the preceding identity implies

$$\frac{(h-1)^{m+1}}{h p_m(h)} \frac{m!}{2\pi i} \int_P \frac{e^{(x+1)z}}{(e^z - h)^2 z^{m+1}}\, dz$$

$$= x s_m^{(0)}(x-1) - \frac{p_{m+1}(h)}{(h-1)\,p_m(h)} s_{m+1}^{(0)}(x-1). \quad (14)$$

If we insert the identity (14) into (11) the reduction formula (12) becomes obvious. ∎

COROLLARY. *Let* $r \geqslant 0$ *denote any integer. For all* $x \in \mathbb{R}$ *and weights* h *satisfying the hypotheses of Theorem* 5 *the relation*

$$s_m^{(r+1)}(x) = \frac{1}{r+1} \left[x s_m^{(r)}(x-1) - \sum_{0 \leqslant l \leqslant r} \frac{1}{l!} \frac{d^l}{dh^l} \left(\frac{p_{m+1}(h)}{(h-1)\,p_m(h)} \right) \right.$$
$$\left. \times \left(s_{m+1}^{(r-l)}(x-1) - s_m^{(r-l)}(x-1)\right) \right] \quad (15)$$

holds.

The identity (15) will be of use in the next section.

5. CONVERGENCE BEHAVIOUR OF CARDINAL EXPONENTIAL SPLINE INTERPOLANTS

Agreeing to retain the above notations, let $h \in \mathbb{C} - (U \cup \mathbb{R}_-)$ and

$$\theta(h) = \sup_{j \in \{-1, +1\}} \left(\left| \frac{z_0(h)}{z_j(h)} \right| \right). \quad (16)$$

Then we have $\theta(h) \in]0, 1[$. From (9) and (5) we conclude that

$$s_m^{(0)}(x) = h^x \left(\sum_{k \in \mathbb{Z}} \frac{e^{2\pi i k x}}{z_k^{m+1}(h)} \right) \Big/ \left(\sum_{k \in \mathbb{Z}} \frac{1}{z_k^{m+1}(h)} \right) \tag{17}$$

$$= h^x \left(1 + \sum_{k \in \mathbb{Z}^\times} (z_0(h)/z_k(h))^{m+1} e^{2\pi i k x} \right) \Big/ \left(1 + \sum_{k \in \mathbb{Z}^\times} (z_0(h)/z_k(h))^{m+1} \right)$$

holds for all $x \in \mathbb{R}$ and all integers $m \geqslant 1$. In particular, there exists a constant $M_0(h) > 0$ so that the estimate

$$|s_m^{(0)}(x) - h^x| \leqslant M_0(h) \, |h|^x \, \theta(h)^m \qquad (x \in \mathbb{R}) \tag{18}$$

holds. In order to establish an estimate of type (18) for cardinal exponential spline interpolants of order $r > 0$ let $d(h) = \inf_{z \in \mathbb{R}_-} |z - h|$ denote the distance in the complex plane $\mathbb{C}$ of the point $h \in \mathbb{C} - (U \cup \mathbb{R}_-)$ from the closed negative real half-line $\mathbb{R}_-$. Define the sequence $(M_r(\cdot, h))_{r \geqslant 0}$ of positive functions on $\mathbb{R}$ by induction on r according to the rules

$$M_0(x, h) = M_0(h),$$

$$M_{r+1}(x, h) = \frac{1}{r+1} \Bigg[|x| \, M_r(x-1, h) + (1 + 2^r) \sum_{0 \leqslant l \leqslant r} \left(2 + \frac{1}{|h-1|^{l+1}} + \frac{1}{d(h)^l} + \frac{|h|}{d(h)^{l+1}} \right) \times |h|^l \, M_{r-l}(x-1, h) \Bigg] \tag{19}$$

whenever $x \in \mathbb{R}$ and $h \in \mathbb{C} - (U \cup \mathbb{R}_-)$. Then (18) admits the following extension:

THEOREM 6. *For all $h \in \mathbb{C} - (U \cup \mathbb{R}_-)$ the cardinal exponential spline interpolants $(s_m^{(r)})_{m \geqslant 1}$ of order $r \geqslant 0$ with respect to the bi-infinite geometric sequence $(h^n)_{n \in \mathbb{Z}}$ satisfy the estimate*

$$\left| s_m^{(r)}(x) - \binom{x}{r} h^{x-r} \right| \leqslant M_r(x, h)(m+1)^r \, |h|^{x-r} \, \theta(h)^m \qquad (x \in \mathbb{R}) \tag{20}$$

for all degrees $m \geqslant 1$.

Proof. In the case $r = 0$ the inequality (20) reduces obviously to the estimate (18). Let us proceed by induction on the order $r \geqslant 0$ and suppose that we have established the estimate (20) for an arbitrary order $r \geqslant 0$. Then we obtain by the Corollary of Theorem 5 for all $x \in \mathbb{R}$

$$\begin{aligned}
&\left| s_m^{(r+1)}(x) - \binom{x}{r+1} h^{x-r-1} \right| \\
&\quad \leqslant \frac{1}{r+1} \left| x \left(s_m^{(r)}(x-1) - \binom{x-1}{r} h^{x-r-1} \right) \right. \\
&\qquad - \sum_{0 \leqslant l \leqslant r} \frac{1}{l!} \frac{d^l}{dh^l} \left(\frac{p_{m+1}(h)}{(h-1)\, p_m(h)} \right) \left(s_{m+1}^{(r-l)}(x-1) \right. \\
&\qquad \left. - \binom{x-1}{r-l} h^{x-r+l-1} \right) - \left. \left(s_m^{(r-l)}(x-1) - \binom{x-1}{r-l} h^{x-r+l-1} \right) \right| \\
&\quad \leqslant \frac{1}{r+1} \left[|x|\, M_r(x-1, h)(m+1)^r \, |h|^{x-r-1} \, \theta(h)^m \right. \\
&\qquad + (1+2^r) \sum_{0 \leqslant l \leqslant r} \frac{1}{l!} \left| \frac{d^l}{dh^l} \left(\frac{p_{m+1}(h)}{(h-1)\, p_m(h)} \right) \right| \\
&\qquad \left. \times M_{r-l}(x-1, h)(m+1)^{r-l} \, |h|^{x-r+l-1} \, \theta(h)^m \right] \\
&\quad \leqslant \frac{1}{r+1} \left[|x|\, M_r(x-1, h) + (1+2^r) \sum_{0 \leqslant l \leqslant r} \frac{1}{l!} \left| \frac{d^l}{dh^l} \left(\frac{p_{m+1}(h)}{(h-1)\, p_m(h)} \right) \right| \right. \\
&\qquad \left. \times |h|^l M_{r-l}(x-1, h) \right] (m+1)^r \, |h|^{x-r-1} \, \theta(h)^m. \qquad (21)
\end{aligned}$$

In order to complete the proof denote by L the logarithmic derivative. In view of the recursion formula (7) we have for $0 \leqslant l \leqslant r$ and $m \geqslant 1$:

$$\frac{1}{l!} \left| \frac{d^l}{dh^l} \left(\frac{p_{m+1}(h)}{(h-1)\, p_m(h)} \right) \right| = \frac{1}{l!} \left| \frac{d^l}{dh^l} \left(\frac{mh+1}{h-1} \right) - hLp_m(h) \right|, \qquad (22)$$

where

$$\frac{1}{l!} \left| \frac{d^l}{dh^l} \left(\frac{mh+1}{h-1} \right) \right| \leqslant (m+1) \left(1 + \frac{1}{|h-1|^{l+1}} \right) \qquad (0 \leqslant l \leqslant r). \quad (23)$$

Moreover, if $\zeta_j \in \mathbb{R}_-$ $(1 \leqslant j \leqslant m-1)$ denote the roots of the Euler–Frobenius polynomial p_m (see Theorem 2 and Corollary 2 of Theorem 3), the following estimate is obtained for $0 \leqslant l \leqslant r$ and $m \geqslant 1$:

$$\frac{1}{l!}\left|\frac{d^l}{dh^l}(hLp_m)(h)\right| = \frac{1}{l!}\left|\frac{d^l}{dh^l}\sum_{1\leqslant j\leqslant m-1}\frac{1}{h-\zeta_j}\right|$$
$$= \frac{1}{l!}\left|\frac{d^l}{dh^l}\left(m-1+\sum_{1\leqslant j\leqslant m-1}\frac{\zeta_j}{h-\zeta_j}\right)\right|$$
$$\leqslant \left|m-1-\sum_{1\leqslant j\leqslant m-1}\frac{1}{(h-\zeta_j)^l}+\sum_{1\leqslant j\leqslant m-1}\frac{h}{(h-\zeta_j)^{l+1}}\right|$$
$$\leqslant (m+1)\left(1+\frac{1}{d(h)^l}+\frac{|h|}{d(h)^{l+1}}\right). \tag{24}$$

Consequently, the left-hand side of (22) admits the estimate

$$\frac{1}{l!}\left|\frac{d^l}{dh^l}\left(\frac{p_{m+1}(h)}{(h-1)\,p_m(h)}\right)\right| \leqslant (m+1)\cdot\left(2+\frac{1}{|h-1|^{l+1}}+\frac{1}{d(h)^l}+\frac{|h|}{d(h)^{l+1}}\right). \tag{25}$$

In view of the definition (19) we obtain by (21)

$$\left|s_m^{(r+1)}(x)-\binom{x}{r+1}h^{x-r-1}\right| \leqslant M_{r+1}(x,h)(m+1)^{r+1}\,|h|^{x-r-1}\,\theta(h)^m. \tag{26}$$

This proves the inequality (20) for all degrees $m\geqslant 1$ and orders $r\geqslant 0$. ∎

In particular, the estimate (20) implies the pointwise convergence theorem of Greville *et al.* [1] for cardinal exponential spline interpolants $(s_m^{(r)})_{m\geqslant 1}$ of order $r\geqslant 0$:

COROLLARY. *Let the weight* $h\in\mathbb{C}-(U\cup\mathbb{R}_-)$ *be given. The convergence*

$$\lim_{m\to\infty} s_m^{(r)}(x)=\binom{x}{r}h^{x-r} \qquad (r\geqslant 0) \tag{27}$$

holds pointwise for all $x\in\mathbb{R}$.

Thus, the pointwise convergence behaviour of the cardinal exponential spline interpolants differs significantly from the pointwise convergence properties of the cardinal logarithmic splines of successively higher degree ("Newman–Schoenberg phenomenon" [3, 6]).

6. SOME GENERAL REMARKS

Let us conclude with some general remarks concerning the application of integral transform techniques to some problems of spline theory. The cardinal exponential splines of degree $m \geqslant 1$ and weight $h \in \mathbb{C}^{\times} - U$, i.e., the elements of the space $\mathfrak{E}_{m,h}(\mathbb{R}; \mathbb{Z}) = \mathbb{C} \cdot S_{m,h}$, are defined on the additive group $\mathbb{R}$ of real numbers. Since the characters $x \rightsquigarrow e^{ixy}$ $(y \in \mathbb{R})$ of the additive group $\mathbb{R}$ form the dual group $\hat{\mathbb{R}}$, the Fourier transform and cotransform are associated with $\mathbb{R}$ in a natural way. As we have pointed out in Section 2, the complex contour integral representation (2) of $S_{m,h}$ arises by an application of the inverse bilateral Laplace transform [4] which is related to the Fourier cotransform of $\mathbb{R}$ by continuing the exponentials $x \rightsquigarrow e^{ixy}$ in the complex y-plane away from the real line $\mathbb{R}$, to reach non-unitary linear representations of the additive group $\mathbb{R}$.

The rôle that the Fourier transform plays for the additive group $\mathbb{R}$ is played by the Mellin transform in the case of the multiplicative group $\mathbb{R}_{+}^{\times}$ of strictly positive real numbers. In [6] we have established a complex contour integral representation of the cardinal logarithmic splines on the open real half-line $\mathbb{R}_{+}^{\times}$ by an application of the inverse Mellin transform. In this case the path of integration is the positively oriented boundary of a suitable closed vertical strip in the complex plane $\mathbb{C}$ that contains the imaginary axis in its interior. Then a procedure similar to those in the previous sections enables us to determine the convergence behaviour of the cardinal logarithmic splines of successively higher degree on the open half-line $\mathbb{R}_{+}^{\times}$.

Finally, let us cast a glance at the case of periodic splines. Among the various classes of univariate spline functions, the theory of cardinal and periodic splines presents nowadays the most complete picture. It is well known that the periodic polynomial splines with respect to equidistant knot sequences on the one-dimensional torus group admit circulant coefficient matrices, i.e., complex coefficient matrices of the form displayed below.

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{pmatrix}. \tag{28}$$

Observe that the entries of the jth row $(2 \leqslant j \leqslant n)$ of the $n \times n$ matrix (28) are obtained from the $(j-1)$st row by shifting all entries one to the right and putting the last entry of row $j-1$ back as the first entry of row j.

Unitary diagonalization of the matrix (28) shows that the circulant matrices are closely related to the finite Fourier cotransform, i.e., to the Fourier cotransform of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ of integers modulo n. In particular, this inter-relation seems to suggest that the finite group $\mathbb{Z}/n\mathbb{Z}$ may control the properties of the univariate periodic splines. However, a deeper look reveals that the genuine group which governs the periodic spline functions is actually the finite Heisenberg group $N(3, \mathbb{Z}/n\mathbb{Z})$ modelled over the ring $\mathbb{Z}/n\mathbb{Z}$ rather than the cyclic group $\mathbb{Z}/n\mathbb{Z}$ itself. The finite nilpotent group $N(3, \mathbb{Z}/n\mathbb{Z})$ forms a subgroup of the special linear group $SL(3, \mathbb{Z}/n\mathbb{Z})$ in three variables over the ring $\mathbb{Z}/n\mathbb{Z}$. Its elements are the 3×3 upper-triangular matrices of the type displayed below

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}, \qquad (29)$$

where the entries x, y, z belong to the ring $\mathbb{Z}/n\mathbb{Z}$. The representation theory of $N(3, \mathbb{Z}/n\mathbb{Z})$ reflects the properties of the finite Fourier cotransform. Moreover, it relates the finite Heisenberg groups with the periodic splines as well as with the theory of attenuation factors of harmonic analysis ("Abminderungsfaktoren" in the terminology used by Quade and Collatz [2]). Due to limitations of space we have to stop our brief preview at this point. For an elaboration of the group theoretic ideas indicated above the reader is referred to a forthcoming monograph on these topics. The notes [7] survey the various results and deal with several applications of nilpotent harmonic analysis to periodic spline functions in terms of the finite Fourier cotransform.

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Printed in Belgium